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# Deterministic exclusion process with a stochastic defect: matrix-product ground states

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**Abstract.** We study a one-dimensional anisotropic exclusion model describing particles moving deterministically on a ring with a single defect, across which they move with probability  $0 < q < 1$ . We show that the stationary state of this model can be represented as a matrix-product state.

## 1. Introduction

The one-dimensional asymmetric exclusion process (ASEP) is a lattice model which describes particles hopping in a preferred direction with stochastic dynamics and hard core exclusion. It has been used to describe various phenomena like growth processes [1, 2], polymerization [3], and traffic flow [4]. Although the ASEP is one of the simplest stochastic many-particle models it shows a rich behaviour ranging from phase transitions [5, 6] to the formation of shocks [7]. During the last few years exact solutions have been found for the stationary state under various boundary conditions. In particular, the ASEP with random sequential dynamics and external particle input and output at the boundaries has been analysed in detail [6, 8–10]. There are only a few exact results for dynamic properties (relaxational behaviour) of the ASEP [11].

A mathematical tool which has proved to be very useful for the study of stochastic lattice models is the so-called matrix-product state (MPS) technique [12–20, 8–10]. Matrix-product states can be understood as a generalization of ordinary factorizable states with a product measure in which numbers are replaced by non-commutative objects. This allows the original problem to be reformulated in terms of algebraic relations. The main advantage of this technique is that once a nontrivial representation of the algebra is available, physical quantities like density–density correlation functions can be computed very easily. In 1993 Derrida *et al* [15] showed that the stationary state of the ASEP with external particle input and output can be written as a MPS. Since then matrix-product methods have been applied to various other problems, e.g. to three-state models [14, 15], excited states in the ASEP [16], certain reaction–diffusion models [17] and models with quasi-parallel updates [18–20].

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Although a lot of work on MPSs has been done, the full range of applications is not yet known<sup>†</sup>.

So far the matrix-product technique has been applied mostly to systems which are homogeneous in the bulk. It is therefore interesting to investigate whether this method can also be applied to systems with impurities. In case of the ASEP two kinds of impurities are discussed in the literature. The first kind of impurities are *defective particles* which jump with a rate lower than that of other particles [15, 22, 23]. Such ‘moving’ impurities can be visualized as slow cars on a motorway which (for sufficiently high particle density) induce traffic jams—a phenomenon which has been related recently to Bose–Einstein condensation [23]. To this problem the matrix-product technique has been applied successfully [15, 22]. The second kind of impurities are *stationary defects* which can be introduced by lowering the hopping rate at specific bonds. As in the previous case, such impurities can provoke the formation of shocks [24]. Defects of this kind play an important role in traffic flow applications where exact solutions are particularly interesting. However, for some reason ASEPs with stationary defects are much harder to solve than systems with moving impurities. Exact results exist only in the case of an exclusion model with parallel dynamics and deterministic hopping in the bulk [25, 26]. The case of stochastic hopping or random sequential dynamics has not yet been solved exactly. Moreover, matrix-product techniques have not been applied to these types of problems.

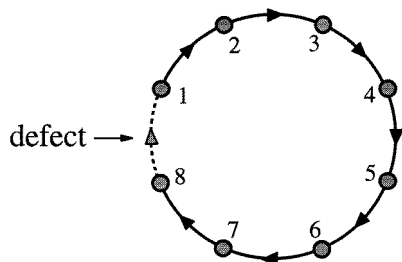
In this paper we consider the stationary state of the ASEP on a ring with various kinds of parallel dynamics, deterministic hopping in the bulk and a single stationary defect. Using Bethe–Ansatz techniques, this problem was first solved in [26] in the case of alternating parallel updates on two sublattices. The solution, however, is fairly complicated and amounts to a list of rules for an operative construction of the stationary state. In the present work we solve the problem in a more compact way by using the matrix-product technique. Moreover, this formalism allows models with different update sequences to be solved within the same framework. In order to demonstrate this, we consider two different update sequences. Our main intention, however, is to outline a method which might help to solve the problem of stochastic hopping in the presence of a defect.

The paper is organized as follows. In section 2 we define the ASEP on a ring with a defect and introduce two different dynamical rules. In section 3 we show that the two-state model on a ring can be reformulated as a four-state model on a linear chain. Using this mapping we formulate a matrix-product ansatz (see section 4) leading to a set of algebraic equations which turns out to be the same for both update sequences. In section 5 we give simple two-dimensional representations of this algebra. Using these representations, some physical quantities are derived and discussed in section 6. Finally in section 7 we summarize our results and discuss possible generalizations.

## 2. The model

The exclusion model we are going to study is defined as follows. Particles of one species move in a clockwise direction on a one-dimensional ring with an even number of sites  $L = 2N$ . Each site can be either free or occupied by one particle. The time evolution is discrete and defined by a certain sequence of parallel updates to be specified below. In the bulk of the chain these updates are deterministic, i.e. in each time step the particles

<sup>†</sup> As it has been shown recently, for every one-dimensional stochastic lattice model with local random sequential dynamics the eigenstates of the Hamiltonian can be written as MPSs [21]. However, matrix representations which are useful for practical purposes seem to be limited to a few models.



**Figure 1.** The ASEP on a ring with  $L = 2N = 8$  sites. In the bulk particles hop deterministically in a clockwise direction. A defect is introduced between sites  $L$  and  $1$  where particles hop with probability  $q$ .

move forward provided that the following site is empty. Between two particular sites (by convention sites  $L$  and  $1$ ) a defective bond is inserted where the particles hop stochastically with a given probability  $0 < q < 1$  (see figure 1).

In what follows we will consider two different update sequences. The first one we call *symmetrized sequential dynamics*: in each time step the first update takes place at the sites  $(N, N + 1)$  just opposite the defect. Then a sequence of pairwise updates follows. These updates are arranged symmetrically with respect to the defect. The first pair to be updated is  $(N - 1, N)$  and  $(N + 1, N + 2)$ , followed by  $(N - 2, N - 1)$  and  $(N + 2, N + 3)$  up to  $(1, 2)$  and  $(L - 1, L)$ . The last pair of sites to be updated is the defect  $(L, 1)$ . As we will see below, this is the simplest dynamical rule in terms of the matrix formalism.

The other dynamical rule we will use is called *sublattice-parallel dynamics* [26]. Here we have to assume that  $N = L/2$  is an even number. Each time step consists of two separate half time steps. In the first half time step particles located at odd sites move one step in a clockwise direction provided that the next site is empty. Then in the second half time step particles at even sites move forward in the same way except for the defect where such moves take place with probability  $q$ . This dynamical rule was introduced in [26] for solving the deterministic ASEP with a defect. A similar dynamics was also used in [25, 18] in the case of the ASEP with external particle input and output.

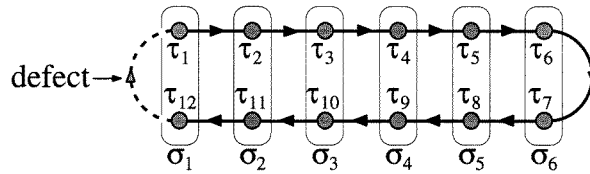
Symmetrized sequential and sublattice-parallel dynamics can be cast in a more formal way as follows. Let  $\tau_j = 0, 1$  be the occupation number at site  $j$  and consider the space of all configurations in a canonical configuration basis. Let  $\mathcal{T}$  ( $\mathcal{T}^{(q)}$ ) be the two-site hopping matrix in the bulk (at the defect):

$$\begin{aligned} \mathcal{T} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \mathcal{T}^{(q)} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 1 - q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{1}$$

Then the transfer matrix  $T_{\text{seq}}$  for symmetrized sequential dynamics reads

$$T_{\text{seq}} = \mathcal{T}_{L,1}^{(q)}(\mathcal{T}_{1,2}\mathcal{T}_{L-1,L})(\mathcal{T}_{2,3}\mathcal{T}_{L-2,L-1}) \dots (\mathcal{T}_{N-1,N}\mathcal{T}_{N+1,N+2})\mathcal{T}_{N,N+1} \tag{2}$$

where the indices indicate pairs of sites on which  $\mathcal{T}$  acts. The transfer matrix for sublattice-



**Figure 2.** By grouping pairs of sites, the two-state ASEP on a ring can be regarded as a four-state model on a linear chain.

parallel updates factorizes into two factors

$$T_{\text{par}} = T_{\text{par}}^{(2)} T_{\text{par}}^{(1)} \quad (3)$$

which are given by

$$\begin{aligned} T_{\text{par}}^{(1)} &= (\mathcal{T}_{1,2} \mathcal{T}_{L-1,L}) (\mathcal{T}_{3,4} \mathcal{T}_{L-3,L-2}) \cdots (\mathcal{T}_{N-1,N} \mathcal{T}_{N+1,N+2}) \\ T_{\text{par}}^{(2)} &= \mathcal{T}_{L,1}^{(q)} (\mathcal{T}_{2,3} \mathcal{T}_{L-2,L-1}) (\mathcal{T}_{4,5} \mathcal{T}_{L-4,L-3}) \cdots (\mathcal{T}_{N-2,N-1} \mathcal{T}_{N+2,N+3}) \mathcal{T}_{N,N+1}. \end{aligned} \quad (4)$$

Notice that both transfer matrices  $T_{\text{par}}$  and  $T_{\text{seq}}$  differ only in the sequence of their (noncommutative) factors. There are, however, many other transfer matrices with different update sequences. As we will see below, the matrix-product formalism allows us to compute ground states for various dynamical rules from the same matrix representation just by rearranging the factors in the matrix product. This is an important advantage of the matrix-product method compared to direct techniques as presented in [26].

### 3. Reformulation as a four-state model

Although the defect breaks translational invariance, the model is still symmetric under reflections with respect to the origin combined with particle-hole symmetry. Since we expect correlations in the stationary state to be subject to this symmetry, it is natural to introduce a matrix ansatz which exploits this symmetry. This can be done by grouping pairs of sites together which are located symmetrically with respect to the defect. This mapping defines a four-state model on a linear chain with  $N$  sites and closed boundaries which is equivalent to the original one (see figure 2). In this language the two-site hopping matrix  $\mathcal{T}_{L,1}^{(q)}$  in the original notation is equivalent to an one-site operator  $\mathcal{L}_1$  at the left boundary acting in a four-dimensional space. Similarly  $\mathcal{T}_{N,N+1}$  corresponds to a one-site operator  $\mathcal{R}_N$  at the right boundary. The block-spin variables  $\sigma_i = (\tau_{L-i+1}, \tau_i)$  can take four different values  $\{0, 1, 2, 3\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ . In this basis the boundary hopping matrices read

$$\begin{aligned} \mathcal{L} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 1-q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \mathcal{R} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (5)$$

In the bulk pairs of opposite hopping, matrices  $(\mathcal{T}_{i,i+1} \mathcal{T}_{L-i,L-i+1})$  are grouped together resulting in a two-site operator  $\mathcal{S}_{i,i+1}$  which is given by a  $16 \times 16$  matrix:

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 1 & 0 & | & 0 & 1 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 & | & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

In terms of these operators, the transfer matrix for sequential dynamics (2) takes the simple form

$$T_{\text{seq}} = \mathcal{L}_1 \mathcal{S}_{1,2} \mathcal{S}_{2,3} \dots \mathcal{S}_{N-1,N} \mathcal{R}_N. \tag{7}$$

Similarly the transfer matrix for symmetrized sublattice-parallel dynamics (4) can be written as

$$T_{\text{par}}^{(1)} = \mathcal{S}_{1,2} \mathcal{S}_{3,4} \dots \mathcal{S}_{N-1,N} \quad T_{\text{par}}^{(2)} = \mathcal{L}_1 \mathcal{S}_{2,3} \mathcal{S}_{4,5} \dots \mathcal{S}_{N-2,N-1} \mathcal{R}_N. \tag{8}$$

So far, we have only rewritten the original model on a ring as a four-state model on a linear chain.

### 4. The matrix-product formalism

As we will show, the four-state formulation allows us to write the stationary state of the asymmetric exclusion model with a defect as a matrix-product state. Let us define noncommutative operators (matrices)  $A_0, A_1, A_2, A_3$  and  $B_0, B_1, B_2, B_3$  as well as bra and ket vectors  $\langle W|$  and  $|V\rangle$  in some auxiliary space and demand that they obey the relations

$$\mathcal{S}(A \otimes B) = (B \otimes A) \tag{9}$$

$$\mathcal{R}A|V\rangle = B|V\rangle \tag{10}$$

$$\langle W|\mathcal{L}B = \langle W|A. \tag{11}$$

where  $A = (A_0, A_1, A_2, A_3)$ ,  $B = (B_0, B_1, B_2, B_3)$  and ‘ $\otimes$ ’ denotes the tensor product in configuration space. This algebra can be used to construct the ground state of the stochastic model we are studying. Let us first consider the case of symmetrized sequential dynamics. As we will show below, the probability  $P_{\text{seq}}(\sigma_1, \dots, \sigma_N)$  to find the stationary system in the configuration  $\{\sigma_1, \dots, \sigma_N\}$  can be written as

$$P_{\text{seq}}(\sigma_1, \dots, \sigma_N) = \frac{1}{Z_N^{\text{seq}}} \langle W|A_{\sigma_1} A_{\sigma_2} \dots A_{\sigma_N}|V\rangle \tag{12}$$

where  $Z_N^{\text{seq}}$  is a normalization constant which is given by

$$Z_N^{\text{seq}} = \langle W|C^N|V \rangle \quad \text{with } C = A_0 + A_1 + A_2 + A_3. \quad (13)$$

For example, the probability to find the four-state chain with  $N = 4$  sites in the configuration  $\{1, 0, 3, 2\} = \{(0, 1), (0, 0), (1, 1), (1, 0)\}$  is given by  $P_{\text{seq}}(1, 0, 3, 2) = \langle W|A_1 A_0 A_3 A_2|V \rangle / \langle W|C^4|V \rangle$ .

The mechanism ensuring the stationarity of these probabilities is precisely the one introduced in [19] and works as follows. Formally, equation (12) may be written as  $|P_{\text{seq}}\rangle = (Z_N^{\text{seq}})^{-1} \langle W|A^{\otimes N}|V \rangle$ . Applying the transfer matrix  $T_{\text{seq}} = \mathcal{L}_1 \mathcal{S}_{1,2} \mathcal{S}_{2,3} \dots \mathcal{S}_{N-1,N} \mathcal{R}_N$  to this state the matrix  $\mathcal{R}$  at the right end of the chain generates the vector  $\mathbf{B} = (B_0, B_1, B_2, B_3)$  at the  $N$ th position in the product  $A^{\otimes N}$  (cf equation (10)). Then, by successively applying  $\mathcal{S}$ , the generated vector  $\mathbf{B}$  is commuted to the left (cf equation (9)). Finally, when reaching the left boundary, the vector  $\mathbf{B}$  is turned into  $\mathbf{A}$  by the action of the defect hopping matrix  $\mathcal{L}$  (cf equation (11)). Consequently, the application of the transfer matrix  $T_{\text{seq}}$  on the state  $|P_{\text{seq}}\rangle$  results in the same state  $|P_{\text{seq}}\rangle$ . Thus the state  $|P_{\text{seq}}\rangle$  is a stationary state of the transfer matrix (7), i.e. of the transfer matrix (2).

For sublattice-parallel dynamics we have to use a slightly different ansatz which involves alternating matrices. We write the stationary state as

$$P_{\text{par}}(\sigma_1, \dots, \sigma_N) = \frac{1}{Z_N^{\text{par}}} \langle W|(A_{\sigma_1} B_{\sigma_2})(A_{\sigma_3} B_{\sigma_4}) \dots (A_{\sigma_{N-1}} B_{\sigma_N})|V \rangle \quad (14)$$

where

$$Z_N^{\text{par}} = \langle W|\left(\sum_{i=1}^4 A_i \sum_{j=1}^4 B_j\right)^{N/2}|V \rangle. \quad (15)$$

For this type of dynamics the mechanism ensuring the stationarity of the probabilities (14) is precisely the one proposed in [18]. Again we use the algebraic relations (9)–(11). Writing  $|P_{\text{par}}\rangle = (Z_N^{\text{par}})^{-1} \langle W|(A \otimes B)^{\otimes N/2}|V \rangle$  one can easily verify that in each half time step the vectors  $\mathbf{A}$  and  $\mathbf{B}$  are exchanged:

$$T_{\text{par}}^{(2)}[(A \otimes B)^{\otimes N/2}] = (B \otimes A)^{\otimes N/2} \quad (16)$$

$$\langle W|T_{\text{par}}^{(1)}[(B \otimes A)^{\otimes N/2}]|V \rangle = \langle W|(A \otimes B)^{\otimes N/2}|V \rangle. \quad (17)$$

Therefore the state  $|P_{\text{par}}\rangle$  is invariant under application of the transfer matrix  $T_{\text{par}}$ .

It should be emphasized that both symmetrized sequential and sublattice-parallel dynamics use the same algebra (9)–(11). One can also consider various other update sequences leading to (9)–(11). For example, if updating is started in the middle of the system and proceeds to both ends from there, the stationary state can be written as  $\langle W|A^{\otimes N} B^{\otimes N}|V \rangle$  with the  $A_i, B_i$  and  $\langle W|, |V \rangle$  obeying (9)–(11). In other words, having determined a nontrivial representation of the algebra given above, one can immediately compute stationary states for various update sequences.

## 5. Representation of the algebra

Equations (9)–(11) define an algebra of eight objects  $A_0, A_1, A_2, A_3, B_0, B_1, B_2, B_3$ . There are 16 bulk equations

$$\begin{aligned} [A_i, B_j] &= 0 & (i = 0 \dots 3) \\ B_0 A_2 &= B_0 A_3 = B_1 A_0 = B_1 A_2 = B_1 A_3 = B_3 A_0 = B_3 A_2 = 0 \\ A_i B_j + A_j B_i &= B_i A_j & (i, j) \in \{(0, 1), (2, 0), (2, 3), (3, 1)\} \\ A_0 B_3 + A_1 B_2 + A_2 B_1 + A_3 B_0 &= B_2 A_1 \end{aligned} \quad (18)$$

four equations at the defect (left boundary of the four-state model):

$$\begin{aligned} \langle W|B_0 = \langle W|A_0 \quad \langle W|(B_1 + qB_2) = \langle W|A_1 \\ \langle W|(1 - q)B_2 = \langle W|A_2 \quad \langle W|B_3 = \langle W|A_3 \end{aligned} \tag{19}$$

and four equations opposite the defect (right boundary):

$$\begin{aligned} A_0|V = B_0|V \quad 0 = B_1|V \\ (A_1 + A_2)|V = B_2|V \quad A_3|V = B_3|V. \end{aligned} \tag{20}$$

Notice that the algebra is invariant under the replacement [27]

$$A_0 \rightarrow \lambda A_0 \quad B_0 \rightarrow \lambda B_0 \quad A_3 \rightarrow \lambda^{-1} A_3 \quad B_3 \rightarrow \lambda^{-1} B_3 \tag{21}$$

where  $\lambda$  is some number.

The algebra (18)–(20) has a complex structure and thus it is practically impossible to determine its representations directly. In order to simplify the problem, we therefore use the following ansatz:

$$B_0 = A_0 \quad B_1 = A_1 - \mathbf{1} \quad B_2 = A_2 + \mathbf{1} \quad B_3 = A_3. \tag{22}$$

A similar ansatz has been used in [19, 20] in order to relate sublattice-parallel and random-sequential updates in the ASEP with external particle input and output. It is obvious that equation (22) imposes strong constraints and thus reduces the space of solutions. However, it turns out that it still includes nontrivial physical representations.

Inserting equation (22) into equations (18)–(20), we obtain a reduced algebra of four objects. It consists of seven bulk equations:

$$A_1 A_0 = A_0 \quad A_1 A_3 = A_3 \quad A_1 A_2 = A_2 \quad A_0 A_2 = A_3 A_2 = A_0 A_3 = A_3 A_0 = 0 \tag{23}$$

and two boundary conditions:

$$\langle W|A_2 = \frac{1 - q}{q} \langle W| \quad A_1|V = |V\rangle. \tag{24}$$

This algebra is much simpler and can be analysed systematically on a computer. In fact, we found the following two-dimensional representation:

$$\begin{aligned} A_0 = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_2 = \frac{1 - q}{q} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad A_3 = 0 \\ \langle W| = (1, 1) \quad |V\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \tag{25}$$

where  $\lambda$  is a free parameter. Algebraically  $\lambda$  is related to the invariance of the algebra in equation (21). Physically it is related to the conservation of the number of particles. In fact, as will be discussed in the next section, the ground states (12) and (14) describe grand-canonical ensembles of systems with different particle numbers where the parameter  $\lambda$  plays the role of a fugacity [27].

As can be verified easily, the canonical ensemble described by the above representation includes *all* sectors with  $M \leq N = L/2$  particles. (If there were more than  $L/2$  particles, at least one of the block spins  $\sigma_i = (\tau_{L-i+1}, \tau_i)$  would be in the state (1,1). Because  $A_3 = 0$  this implies that the corresponding matrix product vanishes.) However, the algebra (23), (24) is invariant under the exchange of the matrices  $A_0 \leftrightarrow A_3$  which immediately gives a second set of representations for systems with more than  $L/2$  particles. Physically this invariance is related to the particle-hole symmetry in the ASEP.



## 6. Some physical quantities for the model with symmetrized sequential dynamics

### 6.1. Micro-canonical ensemble: density-profile for a fixed particle number

In order to derive physical quantities for a fixed particle-number, we have to project the stationary state onto a sector with a specified number of particles  $M$ . A projection formalism for matrix-product states has been introduced recently in [22]. We are now going to apply this formalism to the present model with sequential dynamics (the case of sublattice-parallel dynamics can be treated similarly). Because of particle-hole symmetry, we restrict ourselves to less than half-filled systems where  $M \leq N = L/2$ .

Let  $\chi(m; \sigma_1, \dots, \sigma_n)$  be a function which is 1 if the sites  $1 \dots n$  are occupied by  $m$  particles and 0 otherwise:

$$\chi(m; \sigma_1, \dots, \sigma_n) = \delta\left(n - m + \sum_{i=1}^n (\delta_{\sigma_i,3} - \delta_{\sigma_i,0})\right). \quad (26)$$

Then the probability to find the model with sequential updates and  $M$  particles in the configuration  $\{\sigma_1, \dots, \sigma_N\}$  is given by

$$P_M^{\text{seq}}(\sigma_1, \dots, \sigma_N) = \frac{1}{Z_{N,M}^{\text{seq}}} \chi(M; \sigma_1, \dots, \sigma_N) \langle W | A_{\sigma_1} \dots A_{\sigma_N} | V \rangle \quad (27)$$

where  $Z_{N,M}^{\text{seq}}$  is the normalization constant restricted to the  $M$ -particle sector:

$$Z_{N,M}^{\text{seq}} = \langle W | G_{N,M} | V \rangle \quad (28)$$

$$G_{N,M} = \sum_{\sigma_1, \dots, \sigma_N} \chi(M; \sigma_1, \dots, \sigma_N) A_{\sigma_1} \dots A_{\sigma_N}. \quad (29)$$

The expression  $G_{N,M}$  is the sum of all products of  $N$  matrices with  $M$  particles. By definition, two of these objects can be combined by convolution:

$$G_{N_1+N_2,M} = \sum_{j=0}^M G_{N_1,j} G_{N_2,M-j}. \quad (30)$$

Physical quantities can be expressed in terms of combinations of these objects. For example the density  $c(x)$  to find a particle at site  $x$  in the original model on the ring with  $L = 2N$  sites and  $M \leq N$  particles is given by

$$c(x) = \frac{1}{Z_{N,M}^{\text{seq}}} \begin{cases} \sum_{j=0}^{\min(x-1, M-1)} \langle W | G_{x-1,j} A_1 G_{N-x, M-j-1} | V \rangle & \text{if } x \leq L/2 \\ \sum_{j=0}^{\min(L-x, M-1)} \langle W | G_{L-x,j} A_2 G_{x-N-1, M-j-1} | V \rangle & \text{if } x > L/2. \end{cases} \quad (31)$$

Similar expressions exist for the current and higher correlation functions. Once  $G_{n,m}$  is known, all these quantities can be computed immediately.

The expression  $G_{n,m}$  can be defined recursively by  $G_{n,m} = 0$  if  $m < 0$  or  $m > n$ ,  $G_{0,0} = 1$ , and

$$G_{n,m} = A_0 G_{n-1,m} + (A_1 + A_2) G_{n-1,m-1} + A_3 G_{n-1,m-2}. \quad (32)$$

However, instead of solving this recurrence relation algebraically, it is much simpler to use directly the representation (25). We obtain

$$G_{n,m} = \begin{pmatrix} z_1 & z_3 - z_2 - z_1 \\ 0 & z_2 \end{pmatrix} \quad (33)$$

where

$$z_1 = \delta_{n,m} q^{-n} \quad z_2 = \binom{n}{m} \quad z_3 = 2 \sum_{j=0}^m \binom{n}{m-j} (1-q)^j q^{-j}. \quad (34)$$

Using this result we obtain the normalization (28)

$$Z_{N,M}^{\text{seq}} = 2q^{-M} \sum_{j=0}^M \binom{N}{M-j} q^j q^{M-j} = 2q^{-M} (1-q)^{M-N} I_{1-q}(N-M, M+1) \quad (35)$$

where  $I_z(n, m)$  is the regularized incomplete beta function. Now the particle density profile (31) can be computed easily. Using that  $A_1 = \mathbf{1}$  and  $\langle W | G_{n,m}^{\text{seq}} A_2 = \langle W | \delta_{n,m} (1-q) q^{-n-1}$  we get

$$c(x) = \frac{1}{Z_{N,M}^{\text{seq}}} \begin{cases} Z_{N-1, M-1}^{\text{seq}} & \text{if } x \leq L/2 \\ (1-q) q^{x-2N-1} Z_{x-N-1, M+x-2N-1}^{\text{seq}} & \text{if } x > L/2. \end{cases} \quad (36)$$

This formula holds for less than half-filling  $M \leq N = L/2$  (the case of more than  $L/2$  particles is related by particle-hole symmetry). Writing  $x = L - y + 1$  and keeping the particle density  $\rho = M/N \ll q$  fixed one can now derive the asymptotic expression for the particle density in front of the defect in the thermodynamic limit:

$$\lim_{N \rightarrow \infty} c(L - y + 1) = (1-q) \left( \frac{\rho}{q} \right)^y. \quad (37)$$

A similar result was derived in [26] for sublattice-parallel updating.

### 6.2. Grand-canonical ensemble: correlations for a large lattice

Let us consider a grand-canonical ensemble of systems with  $L \gg 1$  and an average particle number  $\rho L$  where  $\rho$  is some given density. Its probability distribution is given by equation (12) with the matrices (25). The ‘fugacity’ parameter  $\lambda$  has to be chosen such that the particle-density is equal to  $\rho$ , i.e. it has to solve the following equation:

$$2N\rho = \frac{1}{Z_N^{\text{seq}}} \sum_{i=1}^N \langle W | C^{i-1} (A_1 + A_2 + 2A_3) C^{N-i} | V \rangle. \quad (38)$$

As in the previous section we restrict ourselves to particle numbers less than  $N = L/2$  (results for systems with higher particle numbers are obtained exploiting the particle-hole symmetry). Because of  $C = A_0 + A_1 + A_2 + A_3$  and  $A_3 = 0$ , equation (38) can be written as

$$1 - 2\rho = \frac{1}{N Z_N^{\text{seq}}} \sum_{i=1}^N F_{N,i} \quad \text{with } F_{N,i} = \langle W | C^{i-1} A_0 C^{N-i} | V \rangle. \quad (39)$$

The computation of the matrix-elements  $Z_N^{\text{seq}}$  and  $F_{N,i}$  is done most easily in a representation where  $C$  is diagonal. Such a representation can be obtained by means of a similarity transformation from the representation (25). We will use the following matrices and vectors:

$$C = \begin{pmatrix} q^{-1} & 0 \\ 0 & 1 + \lambda \end{pmatrix} \quad A_0 = \lambda \begin{pmatrix} 0 & \frac{2(q-1)}{q+\lambda q-1} \\ 0 & 1 \end{pmatrix} \quad (40)$$

$$\langle W | = \left( 1, \frac{2\lambda q}{q + \lambda q - 1} \right) \quad | V \rangle = \begin{pmatrix} \frac{2(q-1)}{q+\lambda q-1} \\ 1 \end{pmatrix}$$

to compute  $Z_N^{\text{seq}}$  and  $F_{N,i}$  as

$$Z_N^{\text{seq}} = \frac{2}{q + \lambda q - 1} \{ \lambda q (1 + \lambda)^N + (q - 1) q^{-N} \} \tag{41}$$

$$F_{N,i} = \frac{2\lambda}{q + \lambda q - 1} (1 + \lambda)^{N-1} \left\{ \lambda q + \frac{q - 1}{(q(1 + \lambda))^{i-1}} \right\}. \tag{42}$$

The large- $N$  asymptotics of these expressions depend on the magnitudes of the terms  $1 + \lambda$  and  $q^{-1}$ . Let us consider the case  $1 + \lambda > q^{-1}$  with  $1 + \lambda - q^{-1} = O(N^0)$  first. Inserting equations (41) and (42) into equation (39) and approximating for  $N \gg 1$  results in

$$\lambda = \frac{1 - 2\rho}{2\rho}. \tag{43}$$

This equation relates the parameter  $\lambda$  to the density  $\rho$  under the condition  $1 + \lambda > q^{-1}$ , i.e. for

$$\rho < \frac{q}{2}. \tag{44}$$

Equation (43) completes the description of the system for the case  $\rho < q/2$ . One can now easily compute the density-profile or correlation functions. By noticing that  $C$  plays the role of a transfer matrix one reads from equation (40) that all correlations decay exponentially (see e.g. [17]) with a correlation length  $\xi = \{\log(1 + \lambda)q\}^{-1}$ . Using the explicit expression (43) for  $\lambda$  we find

$$\xi \approx \left\{ \log \frac{q}{2\rho} \right\}^{-1} \quad (N \gg 1, \rho < q/2). \tag{45}$$

This result is the same as the one obtained in [26] for the model with sublattice-parallel dynamics.

In the case  $\rho \geq q/2$  the above assumptions on  $\lambda$  do not lead to any result. In order to obtain a finite density we have to assume  $\lambda$  to be of the form

$$\lambda = -1 + q^{-1} \left( 1 + \frac{\lambda'}{N} \right) \tag{46}$$

with some number  $\lambda'$  which is independent of  $N$ . Computing again the large- $N$  asymptotics of equations (41) and (42) and inserting it into equation (39) results in the transcendental equation

$$\frac{1 - 2\rho}{1 - q} = \frac{1 + (\lambda' - 1)e^{\lambda'}}{\lambda'(e^{\lambda'} - 1)} \tag{47}$$

which is soluble for densities  $\rho \geq q/2$ . Equations (46) and (47) relate the parameter  $\lambda$  to the density  $\rho$ . The correlation length, which is given by  $\xi = \{\log(1 + \lambda)q\}^{-1}$  ( $\xi = -\{\log(1 + \lambda)q\}^{-1}$ ) for  $\lambda' > 0$  ( $\lambda' < 0$ ) takes now the form

$$\xi \approx \frac{N}{|\lambda'|} \quad (N \gg 1, \rho \geq q/2) \tag{48}$$

i.e. it is proportional to the system size. Consequently, correlations on a distance  $k \ll N$  decay like  $\lambda'k/N$ . This type of behaviour is completely different from the one observed in the  $\rho < q/2$ -phase. It indicates that the system behaves like the one undergoing sublattice-parallel dynamics [26] where there is a coexistence phase with non-exponential decay of correlations.

### 7. Conclusion

We have shown that MPS techniques can be applied successfully to an asymmetric exclusion processes on a ring with a defect. In order to apply this method, the model on a ring has to be reformulated as a four-state model on a linear chain. This ansatz takes into account the fact that translational invariance is broken by the defect while the system is still symmetric under reflections with respect to the defect combined with particle-hole exchange. The algebra we derived has very simple two-dimensional representations which allow physical quantities like the density profile to be computed directly. Another advantage of the matrix formalism is that the model with different dynamical rules can be solved within the same framework (i.e. with the same algebra and representations).

The ASEP discussed in this paper is a very simple one where particles hop deterministically in the bulk. Our hope is that the present work may show a way to solve the more general problem with probabilistic hopping in the bulk. This generalized model is controlled by three quantities, the bulk hopping rate  $p$ , the defect hopping rate  $q$ , and the particle density  $\rho = M/L$ . In the so-called Hamiltonian limit  $p, q \rightarrow 0, p/q = \text{constant}$ , it includes the case of *random sequential updates* which is an outstanding problem.

When introducing a bulk hopping rate  $p$ , the algebra (18)–(20) has to be replaced by the bulk equations

$$\begin{aligned}
 [A_i, B_i] &= 0 & (i = 1 \dots 4) \\
 (1 - p)A_i B_j &= B_i A_j & (i, j) \in \{(0, 2), (1, 0), (1, 3), (3, 2)\} \\
 A_j B_i + p A_i B_j &= B_j A_i & (i, j) \in \{(0, 2), (1, 0), (1, 3), (3, 2)\} \\
 (1 - p)A_i B_j + p(1 - p)A_1 B_2 &= B_i A_j & (i, j) \in \{(0, 3), (3, 0)\} \\
 (1 - p)^2 A_1 B_2 &= B_1 A_2 \\
 A_2 B_1 + p A_0 B_3 + p^2 A_1 B_2 + p A_3 B_0 &= B_2 A_1
 \end{aligned} \tag{49}$$

and the boundary conditions

$$\begin{aligned}
 \langle W|B_0 = \langle W|A_0 & \quad A_0|V \rangle = B_0|V \rangle \\
 \langle W|(B_1 + q B_2) = \langle W|A_1 & \quad (1 - p)A_1|V \rangle = B_1|V \rangle \\
 \langle W|(1 - q)B_2 = \langle W|A_2 & \quad (pA_1 + A_2)|V \rangle = B_2|V \rangle \\
 \langle W|B_3 = \langle W|A_3 & \quad A_3|V \rangle = B_3|V \rangle.
 \end{aligned} \tag{50}$$

This algebra is even more complicated than the previous one and we were not able to find representations or to prove its consistency. However, following the ideas of [19], one could again assume the additional relations given in equation (22) to hold. This reduces the algebra to

$$\begin{aligned}
 p A_1 A_0 = A_0 & \quad p A_1 A_3 = A_3 & \quad p A_0 A_2 = (1 - p)A_0 & \quad p A_3 A_2 = (1 - p)A_3 \\
 A_0 A_3 = A_3 A_0 = \frac{1 - p}{p(2 - p)}(A_1 + A_2) & \quad A_1 A_2 = \frac{(1 - p)^2}{p(2 - p)}A_1 + \frac{1}{p(2 - p)}A_2
 \end{aligned} \tag{51}$$

together with the boundary equations

$$\langle W|A_2 = \frac{1 - q}{q} \langle W| & \quad A_1|V \rangle = \frac{1}{p}|V \rangle. \tag{52}$$

Likewise, although this algebra is much simpler, we were not able to prove its consistency and the existence of nontrivial representations. As usual in ASEPs including a Hamiltonian limit, such nontrivial representations are expected to be infinite-dimensional. The ASEP with a defect and random sequential updates remains as an open problem.

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